Meeting 4 | Math 22

Section 9.4.2 The Length of $u \times v$

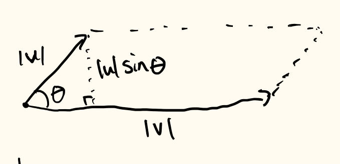
You can compute the length of the cross product by apply the definition of magnitude to compute $|u\times v|^2$

The Length of $u \times v$ Let $u, v - \mathbb{R}^3$. Then

$|u\times v| = |u||v|\sin\theta$

where $0 \le \theta \le \pi$ is the angle tween u and v.

looks like $u\cdot v = |u||v|\cos(\theta)$.



From the picture:

$|u\times v| = |u||v|\sin\theta$
= area of parallelogram spanned by u, v.

This explains why $|u\times v| = 0 \iff u, v$ are parallel.

Consider $u = \langle u_1, u_2, 0\rangle$, $v = \langle v_1, v_2, 0\rangle$

$$|\hat{k}|\begin{vmatrix} u_1 & u_2 \\ v_1 & v_2\end{vmatrix} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ u_1 & u_2 & 0 \\ v_1 & v_2 & 0\end{vmatrix} = |u\times v| = \text{area of parallelogram}$$

Thus, $\begin{vmatrix} u_1 & u_2 \\ v_1 & v_2\end{vmatrix}$ is the (signed) area of a parallelogram!

Activity 9.4.3

- Complete Activity 9.4.3 and discuss w/ your group.
- Class discussion.

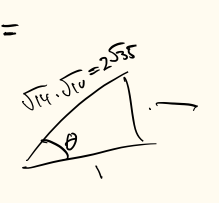
(a) $|u\times v| = |u||v|\sin\theta$

$\theta = \cos^{-1}\left(\frac{u\cdot v}{|u||v|}\right) = \cos^{-1}\left(\frac{1}{\sqrt{14}\cdot\sqrt{10}}\right)$

$= \sqrt{14}\cdot\sqrt{10}\,\sin$

$= \sqrt{14}\cdot\sqrt{10}\,\sin\left(\cos^{-1}\left(\frac{1}{\sqrt{14}\cdot\sqrt{10}}\right)\right) =$

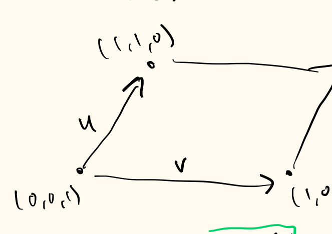
$= \sqrt{14}\cdot\sqrt{10}\ \ldots$



Too much work

$= \sqrt{139}$

(b)



$u = \langle 1, 1, -1\rangle$
$v = \langle 1, 0, 0\rangle$

$$u\times v = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 1 & -1 \\ 1 & 0 & 0\end{vmatrix} = \begin{vmatrix} \hat{j} & \hat{k} \\ 1 & -1\end{vmatrix} = -\hat{j} - \hat{k} = \langle 0, -1, -1\rangle$$

So $|u\times v| = \sqrt{2}$

Section 9.4.3 The Direction of $u\times v$

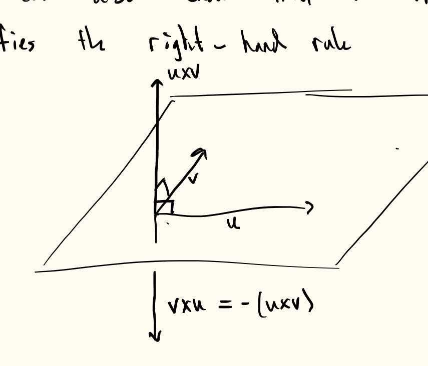
The claim is that $u\times v$ is perp. to both u and v. Set $u = \langle u_1, u_2, u_3\rangle$
$v = \langle v_1, v_2, v_3\rangle$

$$u\cdot(u\times v) = u\cdot\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3\end{vmatrix} = \begin{vmatrix} u_1 & u_2 & u_3 \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3\end{vmatrix} = 0$$

The determinant of a matrix with 2 of the same row or column is always 0.

$v\cdot(u\times v) = v\cdot(-v\times u)$
$= -(v\cdot(v\times u))$
$= 0$ by above.

You can also show that the triple $(u, v, u\times v)$ satisfies the right-hand rule

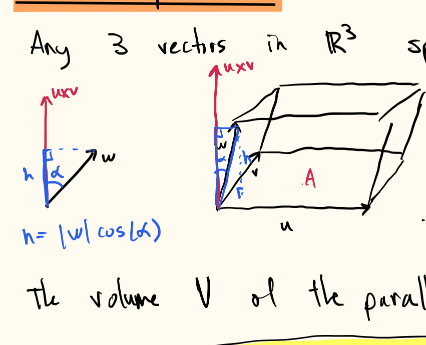


Geometric Definition of $u\times v$ Let $u, v \in \mathbb{R}^3$. Then $u\times v$ is the unique vector satisfying:

1. $u\times v$ is perpendicular to both u and v and $(u, v, u\times v)$ satisfies the right hand rule. (direction)
2. $|u\times v| = |u||v|\sin\theta$ (magnitude) = area of parallelogram spanned by u & v where $0 \le \theta \le \pi$ is the angle between u & v.

Scalar Triple Product

Any 3 vectors in $\mathbb{R}^3$ span a parallelepiped



The volume V of the parallelepiped is:

$V = Ah = |u\times v|\,|w|\cos(\alpha) = |w\cdot(u\times v)|$

The scalar $w\cdot(u\times v)$ is called the scalar triple product of w, u, v.

If $w = \langle w_1, w_2, w_3\rangle$, $u = \langle u_1, u_2, u_3\rangle$, $v = \langle v_1, v_2, v_3\rangle$, then

$$w\cdot(u\times v) = \begin{vmatrix} w_1 & w_2 & w_3 \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3\end{vmatrix}$$

Activity 9.4.4

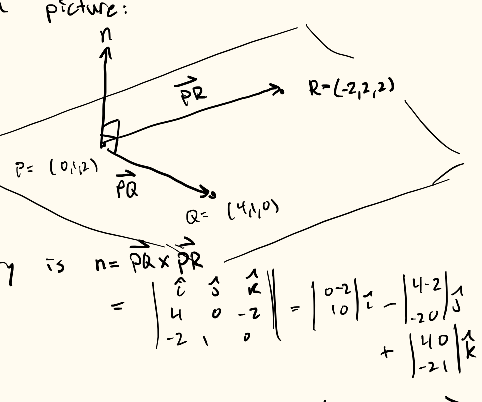
- Complete Activity 9.4.4 and discuss w/ your group.
- Class discussion.

(a) Compute $u\times v$ and $|u\times v|$. Then the vectors are $\frac{u\times v}{|u\times v|}$ and $-\frac{u\times v}{|u\times v|}$.

(b) Use scalar triple product:

$$|w\cdot(u\times v)| = \begin{vmatrix} 3 & 2 & 1 \\ 3 & 5 & -1 \\ 2 & -2 & 1\end{vmatrix} = \left|3\begin{vmatrix} 5 & -1 \\ -2 & 1\end{vmatrix} - 3\begin{vmatrix} 3 & -1 \\ 2 & 1\end{vmatrix} + \begin{vmatrix} 3 & 5 \\ 2 & -2\end{vmatrix}\right| = |3\cdot3 - 3\cdot5 - 16| = 22$$

(c) Draw a picture:



One possibility is $n = \vec{PQ}\times\vec{PR}$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 4 & 0 & -2 \\ -2 & 1 & 0\end{vmatrix} = \begin{vmatrix} 0 & -2 \\ 1 & 0\end{vmatrix}\hat{i} - \begin{vmatrix} 4 & -2 \\ -2 & 0\end{vmatrix}\hat{j} + \begin{vmatrix} 4 & 0 \\ -2 & 1\end{vmatrix}\hat{k} = \langle 2, 4, 4\rangle$$

(d) Right-hand rule!

(e) The vectors lie in the same plane if and only if the volume of the parallelepiped they span is zero. Compute

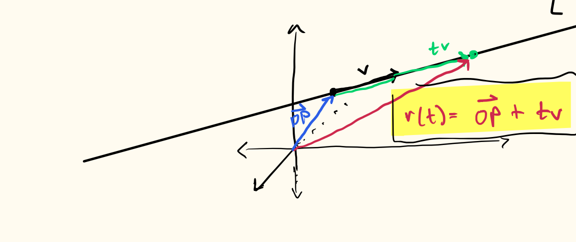
$$c\cdot(a\times b) = \begin{vmatrix} 0 & 1 & 0 \\ 1 & 3 & -2 \\ 2 & 1 & -4\end{vmatrix} = -\begin{vmatrix} 1 & -2 \\ 2 & 4\end{vmatrix} = 0.$$

The vectors are coplanar.

End of Section 9.4.

Section 9.5.1 Lines and Planes

Definition 9.5.3 A *line* through a point P in the direction of a vector v is the set of terminal points of all vectors parallel to v with initial point P.



Vector form of a Line The *vector form* of the line through P in the direction of v is the vector-valued function

$r(t) = \vec{OP} + tv$.

Reading Debrief

- Discuss Activity 9.5.2 w/ your group.
- Questions about anything from Section 9.5.1?

Section 9.5.2 Parametric Equations of a Line

Suppose $r(t) = r_0 + tv$ is a line thru $r_0 = \langle x_0, y_0, z_0\rangle$ w/ direction $v = \langle a, b, c\rangle$

Then

$r(t) = \langle x_0, y_0, z_0\rangle + t\langle a, b, c\rangle$
$= \langle x_0 + ta, y_0 + tb, z_0 + tc\rangle$

The linear functions $x(t) = x_0 + ta$
$y(t) = y_0 + tb$
$z(t) = z_0 + tc$

specify the x, y, z components of a point on the line at time t.

Parametric Eq of a Line The functions

$x(t) = x_0 + at$ $y(t) = y_0 + bt$ $z(t) = z_0 + ct$

are called *the parametric eq's of a line*.

Activity 9.5.3

- Complete 9.5.3 and discuss w/ your group
- Class discussion.

(a) From 9.5.2, $r(t)$ and $v(s)$ are the same line. Using $s(t) = \langle -5 + 6t, 2t, -3 + 2t\rangle$, one possible set of parametric equations is

① $x(t) = -5 + 6t$
② $y(t) = 2t$
③ $z(t) = -3 + 2t$

(b) No. If $(1, 2, 1)$ lies on L, then eq. ② implies that $t = 1$. But $z(1) = -3 + 2\cdot1 = -1$.

(c) The direction is $\langle 4, -3, 2\rangle$.

(d) If they do, then there is a solution to the system of equations

$$\begin{cases} -5 + 6t = 11 + 4s & R_1 \\ -2t = 1 - 3s & R_2 \\ -3 + 2t = 3 + 2s & R_3\end{cases}$$

Linear algebraic method for solving system:

R_1 $6t - 4s = 16$
R_2 $-2t + 3s = 1$
R_3 $2t - 2s = 6$

Row-reduction algorithm:

$$\begin{bmatrix} 6 & -4 & 16 \\ -2 & 3 & 1 \\ 2 & -2 & 6\end{bmatrix} \xrightarrow{R_2 + R_3 \to R_3} \begin{bmatrix} 6 & -4 & 16 \\ -2 & 3 & 1 \\ 0 & 1 & 7\end{bmatrix} \xrightarrow{R_2 - 3R_3 \to R_2} \begin{bmatrix} 6 & -4 & 16 \\ -2 & 0 & -20 \\ 0 & 1 & 7\end{bmatrix}$$

$$\xrightarrow{-\frac{1}{2}R_2 \to R_2} \begin{bmatrix} 6 & -4 & 16 \\ 1 & 0 & 10 \\ 0 & 1 & 7\end{bmatrix} \xrightarrow{R_1 - 6R_2 \to R_1} \begin{bmatrix} 0 & -4 & -44 \\ 1 & 0 & 10 \\ 0 & 1 & 7\end{bmatrix}$$

$$\xrightarrow{R_1 + 4R_3 \to R_1} \begin{bmatrix} 0 & 0 & -16 \\ 1 & 0 & 10 \\ 0 & 1 & 7\end{bmatrix}$$

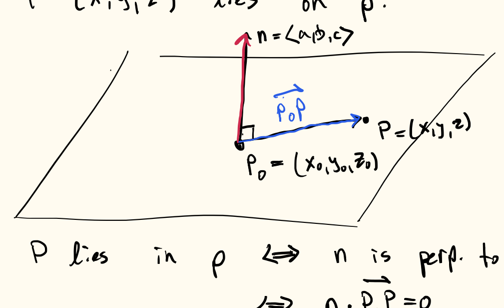
This says $0 = -16$. Nonsense!

By using substitution or linear algebra, there is no solution and the lines do not intersect.

Section 9.5.3 Planes in Space

Definition 9.5.6 A *plane* p through a point P_0 with normal vector n is the set of terminal points of all vectors with initial point P_0 that are perpendicular to n.

How can we determine when an arbitrary point $P = (x, y, z)$ lies on p?



Notice: P lies in $p \iff n$ is perp. to $\vec{P_0P}$
$\iff n\cdot\vec{P_0P} = 0$
$\iff \langle a, b, c\rangle\cdot\langle x - x_0, y - y_0, z - z_0\rangle = 0$
$\iff a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$

Equations of a Plane Let p be a plane through $P_0 = (x_0, y_0, z_0)$ with normal vector $n = \langle a, b, c\rangle$.

1. The *scalar equation* of p is given by $a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$.
2. Let $P = (x, y, z)$ be an arbitrary point. The *vector equation* of p is given by $n\cdot\vec{P_0P} = 0$.

The scalar equation ① can also be written as

$ax + by + cz = d$

where $d = ax_0 + by_0 + cz_0$.

Example $3x + 5y + 7z = 11$ is a plane w/ normal vector $\langle 3, 5, 7\rangle$. A point on the plane is $(0, 0, 11/7)$.

Activity 9.5.4

- Complete 9.5.4 and discuss w/ your group
- Class discussion.

(a) An equation is

$2(x - 0) - 1(y - 2) + 1\cdot(z - 4) = 0$.

Can combine the constants to get

$2x - y + z = 2$.

(b) No, because $(2, 0, 2)$ does not satisfy the eq: $2\cdot2 - 0 + 2 = 4 \ne 2$.